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Effective effective interactions

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Abstract. The use of effective field theory techniques allows the basic physics to be extracted from otherwise complex systems. We cite examples from quantum mechanics, condensed-matter physics, QCD, and quantum gravity.

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1 Introduction

There exist many situations in physics wherein two quite separate scales —one heavy and one light— are involved. In such cases, provided that one is working at an energy momentum small compared to the heavy scale, then one can write the interaction in terms of an "effective" Hamiltonian which is written only in terms of the light degrees of freedom but which fully incorporates all (virtual) effects associated with the heavy scale [1]. Such an effective interaction is often able to isolate and more clearly represent the underlying physics of a given process than is a rigorous treatment involving the entire system. A simple example from classical mechanics involves the use of the effective gravitational potential

$$V = mg(r - R_{\mathrm{E}})$$

in the vicinity of the earth's surface rather than the full Newtonian form

$$V = -\frac{GmM_{\mathrm{E}}}{r}$$

which is valid everywhere [2]. We give additional simple examples below from the realms of quantum mechanics, condensed-matter physics, QCD, and quantum gravity.

2 Examples

2.1 Quantum mechanics

A nice example of effective field theory (EFT) methods is found in quantum mechanics, from the classic problem of understanding why the sky is blue [3]. The answer, of course, comes from Compton scattering of visible light from the atoms which make up the earth's atmosphere. First, however, we outline the full analysis. Using the simple Hamiltonian [4]

$$H = \frac{(\boldsymbol{p} - e\boldsymbol{A})^2}{2m} + e\phi \tag{1}$$

and lowest-order perturbation theory, one finds the Kramers-Heisenberg amplitude

$$\begin{aligned} \mathrm{Amp} = -\frac{e^2}{m\sqrt{2\omega_i\omega_f}} \Bigg[& \hat{\epsilon}_i \cdot \hat{\epsilon}_f \\ & + \frac{1}{m}\sum_n \Bigg(\frac{\hat{\epsilon}_f \cdot \langle 0|\boldsymbol{p}e^{-i\boldsymbol{q}_f\cdot\boldsymbol{r}}|n\rangle\langle n|\boldsymbol{p}e^{i\boldsymbol{q}_i\cdot\boldsymbol{r}}|0\rangle \cdot \hat{\epsilon}_i}{\omega_i + E_0 - E_n} \\ & + \frac{\hat{\epsilon}_i \cdot \langle 0|\boldsymbol{p}e^{i\boldsymbol{q}_i\cdot\boldsymbol{r}}|n\rangle\langle n|\boldsymbol{p}e^{-i\boldsymbol{q}_f\cdot\boldsymbol{r}}|0\rangle \cdot \hat{\epsilon}_f}{E_0 - \omega_f - E_n} \Bigg) \Bigg] . \end{aligned} \tag{2}$$

When the energy of the photons is small compared to a typical excitation energy, the use of simple quantum-mechanical identities allows one to simplify eq. (2) to the form

$$\mathrm{Amp} \propto \omega^2 \,,$$

where the constant of proportionality involves the electric polarizability of the atom. The corresponding cross-section is proportional to ω^4 and explains why the sky is blue.

A corresponding EFT discussion is much simpler. An effective interaction which describes Compton scattering from an atomic system must satisfy certain basic strictures. It must be quadratic in the vector potential $\boldsymbol{A}$, must be a rotational scalar, be gauge invariant, and be symmetric under spatial inversion and time reversal. After a little thought it becomes clear that the lowest order effective Hamiltonian must have the form

$$H_{\mathrm{eff}} = \frac{1}{2}4\pi\alpha_{\mathrm{E}}\boldsymbol{E}^2 + \frac{1}{2}4\pi\beta_M\boldsymbol{H}^2 \,. \tag{3}$$

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Since $\boldsymbol{E}, \boldsymbol{H} \sim \omega\hat{\epsilon}$ we easily find the desired form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \propto |\langle f|H_{\mathrm{eff}}|i\rangle|^2 \propto \omega^4 \,. \tag{4}$$

Obviously then EFT has isolated the basic physics with a minimum of formalism.

2.2 Superconductivity

A second example, this time from condensed-matter physics, is that of superconductivity. In this case electron-lattice interactions lead, in a complex fashion, to the effective BCS Hamiltonian, which involves an effective attractive interaction between electron pairs which are anticorrelated in spin and momentum [5],

$$\begin{aligned} H_{\mathrm{BCS}} = &\sum_{\boldsymbol{k},s} \psi^\dagger_{\boldsymbol{k},s}\left(i\frac{\partial}{\partial t} - \frac{\boldsymbol{k}^2}{2m} - \mu\right)\psi_{\boldsymbol{k},s} \\ &-g\sum_{\boldsymbol{k}} \psi^\dagger_{\boldsymbol{k},\uparrow}\psi^\dagger_{-\boldsymbol{k},\downarrow}\psi_{-\boldsymbol{k},\downarrow}\psi_{\boldsymbol{k},\uparrow} \,. \end{aligned} \tag{5}$$

However, while the physics of superconductivity can be found by a detailed many-body solution of eq. (5) [6], it is certainly not apparent from the form of the Hamiltonian. Nevertheless, one can spotlight the essential physics by rewriting the BCS Hamiltonian in terms of the "right" degrees of freedom —the anticorrelated electron pairs. Defining

$$\phi = \psi_{q\uparrow}\psi_{-q\downarrow}$$

and rewriting eq. (5) in terms of ϕ at fixed temperature T, via imaginary-time methods, yields the familiar Landau-Ginzburg Hamiltonian [7]

$$\begin{aligned} H_{\mathrm{eff}} = &c(T)\phi^\dagger\frac{(-i\boldsymbol{\nabla} - e^*\boldsymbol{A})^2}{2m^*} \\ &+a(T)\phi^\dagger\phi + b(T)(\phi^\dagger\phi)^2 + \dots \,, \end{aligned} \tag{6}$$

where $e^* = 2e$, $m^* = 2m$. Here $b(T), c(T)$ are positive monotonic functions of temperature, but [8]

$$a(T) \propto \ln\frac{T}{T_{\mathrm{c}}}$$

and changes sign at a critical temperature T_{c}. Then when $T > T_{\mathrm{c}}$, the effective potential has a single-well shape with the minimum at $\phi^\dagger\phi = 0$. However, when $T < T_{\mathrm{c}}$ the effective potential has a double well, with a minimum at $\phi^\dagger\phi \neq 0$, corresponding to the superconducting phase. Here too then the use of EFT methods has allowed the extraction of the basic physics of the system.

2.3 QCD

A particularly nice example of the use of EFT in particle/nuclear physics is the utilization of chiral perturbation theory in order to understand low-energy QCD [9]. Since such methods will be extensively covered in parallel session, I will not give a detailed discussion here. However, it is worth pointing out that there exist in this case interesting parallels with the superconducting discussion given above. As in superconductivity, we begin with an interaction:

$$\mathcal{L}_{\mathrm{QCD}} = \bar{q}(i\not{D} - m)q - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} \,, \tag{7}$$

which is difficult (impossible!) to solve. However, at low energies we can make contact between theory and experiment by rewriting eq. (7) in terms of new variables

$$\phi = \psi_{q\uparrow}\bar{\psi}_{-q\downarrow}$$

and the nonlinear quantity

$$U = \exp\left(\frac{i}{F_\pi}\sum_i \lambda_i\phi_i\right)$$

which are the pseudoscalar mesons. What results is a lowest-order effective interaction:

$$\mathcal{L}_{\mathrm{eff}} = \frac{F_\pi^2}{4}\mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger) + \frac{F_\pi^2}{4}\mathrm{Tr}(B_0 m(U + U^\dagger)) \tag{8}$$

in terms of the physical degrees of freedom —π, K, η— rather than in terms of the quarks and gluons in terms of which eq. (7) is written. This effective chiral Lagrangian encodes the underlying symmetries of QCD and has had enormous phenomenological success [10].

2.4 Quantum gravity

I will close with a brief discussion of a subject that folklore says is impossible —quantum gravity. I agree that we do not know how to represent such a theory at all scales, but I want to show you that provided that we treat things as an EFT and only try to represent the long-range aspects, then quantum gravity can be made fully consistent [11]. In order to demonstrate this, we have recently calculated the radiation (photon loop) corrections to the energy-momentum tensor, which describes the coupling of a charged point mass to the gravitational field. In the case of a spin-(1/2) system, the most general form of such a matrix element is [12]

$$\begin{aligned} \langle p'|T_{\mu\nu}|p\rangle = \bar{u}(p')\Big[&F_1(q^2)P_\mu P_\nu \\ &-F_2(q^2)\frac{i}{4M}\left(\sigma_{\mu\lambda}q^\lambda P_\nu + \sigma_{\nu\lambda}q^\lambda P_\mu\right) \\ &+F_3(q^2)\frac{1}{4M^2}(q_\mu q_\nu - q^2\eta_{\mu\nu})\Big] \,, \end{aligned} \tag{9}$$

where $P_\mu = \frac{1}{2}(p' + p)_\mu$, $q_\mu = (p - p')_\mu$ and we have calculated each of these form factors. The general form of each is found to be [12,13]

$$\begin{aligned} F_i(q^2) = 1 + \frac{\alpha}{\pi}\Big(&a\frac{q^2}{M^2}\sqrt{\frac{M^2}{-q^2}} + b\frac{q^2}{M^2}\ln \\ &-\frac{q^2}{M^2} + c\frac{q^2}{M^2} + \dots\Big) \,. \end{aligned} \tag{10}$$

The surprising thing about these forms is the appearance of the nonanalytic terms $\propto \sqrt{-q^2}, q^2 \ln -q^2$, for which the usual structure measure

$$\langle r^2 \rangle = \frac{6}{F(q^2)} \frac{\mathrm{d}F(q^2)}{\mathrm{d}q^2}\bigg|_{q^2=0} = \infty \, .$$

The meaning of these forms can be found ty taking the Fourier transform in order to go to coordinate space [14]

$$T_{\mu\nu}(\boldsymbol{r}) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} e^{-i\boldsymbol{q}\cdot\boldsymbol{r}} T_{\mu\nu}(q) = \frac{GM}{r^2} + a\frac{G\alpha}{r^4} + b\frac{G\alpha\hbar}{Mr^5} + \dots \, . \tag{11}$$

The meaning of such terms is clear. The classical correction $\propto \frac{G\alpha}{r^4}$ arises from the energy momentum of the electric/magnetic fields which extend outside the charged mass [15]. Indeed, the energy-momentum tensor for the electromagnetic field is given by

$$T^{EM}_{\mu\nu} = -F_{\mu\lambda}F_\nu{}^\lambda + \frac{1}{4}F_{\lambda\delta}F^{\lambda\delta} \, . \tag{12}$$

Using the Dirac value —two— for the gyromagnetic ratio, this yields forms

$$\begin{aligned} T^{EM}_{00}(\boldsymbol{r}) &= \frac{1}{2}\boldsymbol{E}^2 = \frac{\alpha}{8\pi r^4} \, , \\ T^{EM}_{0i}(\boldsymbol{r}) &= -(\boldsymbol{E}\times\boldsymbol{B})_i = -\frac{\alpha}{4\pi m r^6}(\boldsymbol{S}\times\boldsymbol{r})_i \, , \\ T^{EM}_{ij}(\boldsymbol{r}) &= -E_iE_j + \frac{1}{2}\delta_{ij}\boldsymbol{E}^2 = \\ &\quad -\frac{\alpha}{4\pi r^4}\left(\frac{r_ir_j}{r^2} - \frac{1}{2}\delta_{ij}\right) , \end{aligned} \tag{13}$$

which agree precisely with the classical piece of eq. (11), while the quantum corrections $\propto \frac{G\alpha\hbar}{Mr^5}$ can be understood in terms of the feature that a classical distance must be modified in going to quantum mechanics by an uncertainty of order the Compton wavelength. Then

$$r \to r + \frac{\hbar}{m}$$

and

$$\frac{1}{r^4} \to \frac{1}{r^4} - \frac{\hbar}{mr^5} + \dots \, ,$$

so that the quantum pieces of eq. (11) are seen to be zitterbewegung modifications to the distance scale r [16].

Alternatively, one can use the Einstein equation

$$\Box h_{\mu\nu} = -16\pi G(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T) \tag{14}$$

to convert these modifications of $T_{\mu\nu}$ to changes in the metric, leading to forms

$$h_{\mu\nu} \sim \frac{GM}{r} + a\frac{G\alpha}{r^2} + b\frac{G\alpha\hbar}{Mr^3} + \dots \, . \tag{15}$$

Here the classical corrections

$$\begin{aligned} h^{cl}_{00}(\boldsymbol{r}) &= \frac{G\alpha}{r^2} \, , \\ h^{cl}_{0i}(\boldsymbol{r}) &= -\frac{G\alpha}{mr^4}(\boldsymbol{S}\times\boldsymbol{r})_i \, , \\ h^{cl}_{ij}(\boldsymbol{r}) &= \frac{G\alpha}{r^4}r_ir_j \end{aligned} \tag{16}$$

are again well known and agree precisely with the Kerr-Newman metric, which describes a rotating massive charge [17]. However, we also find unique quantum corrections which of necessity go along with this classical form.

We see then that when treated as an effective field theory, quantum gravity is well defined and leads to specific long-distance corrections to the lowest-order forms. Such modifications are required and follow simply from the feature that electromagnetism and gravity must be described in terms of both classical and quantum mechanics.

In closing, I want to mention that we are currently in the process and calculating the corresponding *gravitational* loop corrections to the gravitational couplings [18]. In this case, the metric corrections from classical loop contributions are higher order in G and can be shown to be identical to the $\mathcal{O}(G^2)$ corrections to the classical Kerr-Newman metric, and there are corresponding and unique quantum-mechanical corrections of $\mathcal{O}(G^2\hbar/m)$. Again, however, when interpreted in terms of effective field theory, quantum gravity contains completely consistent and understandable forms in the long distance regime.

3 Conclusions

We have shown above, in a variety of circumstances,

i) Quantum Mechanics,
ii) Superconductivity,
iii) Quantum Chromodynamics,
iv) Quantum Gravity,

how the use of EFT methods allows extraction of the relevant physics in a fully consistent and efficient fashion. Such methods truly are effective!

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References

1. See, *e.g.*, A. Manohar, hep-ph/9606222; D. Kaplan, nucl-th/9506035; H. Georgi, Ann. Rev. Nucl. Sci. **43**, 209 (1995).
2. D. Phillips, nucl-th/0203040.
3. B.R. Holstein, Am. J. Phys. **67**, 422 (1999).
4. See, *e.g.*, J.J. Sakurai *Advanced Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1967).
5. J. Bardeen, L.N. Cooper, J.R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

6. S. Weinberg *Quantum Field Theory*, Vol. **II** (Cambridge University Press, Cambridge 1996)
7. V.L. Ginzburg, L.D. Landau, Sov. Phys. JETP **20**, 1064 (1950).
8. L.P. Gorkov, Sov. Phys. JETP **7**, 505 (1959).
9. J. Gasser, H. Leutwyler, Ann. Phys. (N.Y.) **158**, 142 (1984); Nucl. Phys. B **250**, 465 (1985).
10. See, *e.g.*, B.R. Holstein, Int. J. Mod. Phys. A **7**, 7873 (1992).
11. J.F. Donoghue, Phys. Rev. D **50**, 3974 (1994).
12. F. Berends, R. Gastmans, Phys. Lett. B **55**, 311 (1975) and Ann. Phys. (N.Y.) **98**, 225 (1976).
13. K. Milton, Phys. Rev. D **15**, 538 (1977).
14. J.F. Donoghue, B.R. Holstein, B. Garbrecht, T. Konstandin, Phys. Lett. B **529**, 132 (2002).
15. See, *e.g.* J.D. Jackson, *Classical Electrodynamics* (Wiley, New York 1975).
16. See, *e.g.*, B.R. Holstein, *Topics in Advanced Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1992).
17. R. Kerr, Phys. Rev. Lett. **11**, 237 (1963); E.T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, R. Torrence, J. Math. Phys. **6**, 918 (1965).
18. N.E.J. Bjerrum-Bohr, J.F. Donoghue, B.R. Holstein, in preparation.